Math Logic: Model Theory & Computability
Lecture 06

In order to avoid future headache with the definition of free variables in a formula, we make the following convention to exclude formulas of the form $(\forall x(x=y)) \vee (\neg(x=z))$, where the same variable has both quantified and free occurances.

Convention. A variable v_i is said to be *quantified* in a σ-formula φ if there is a subformula of the form $\exists v_i \psi$ in φ. We call φ *valid* if each variable v_i is quantified in it at most once, i.e. occurs in at most one subformula of the form $\exists v_i \psi$, and if it has such an occurance, then it doesn't appear anywhere else outside of $\exists v_i \psi$.

Thus, $(\overset{\neg\exists x\neg}{\forall x}(x=y)) \vee (\neg(x=z))$ is invalid because x is quantified but also appears outside of the subformula $\exists x \neg(x=y)$. The only inconvenience with this convention is when we use binary logical connectives such as $\varphi \vee \psi$, $\varphi \wedge \psi$, $\varphi \to \psi$, we need to make sure that φ and ψ use different sets of quantified variables. Below when we say a "σ-formula" we mean "valid σ-formula".

Def. A variable v_i is said to be *free* in a (valid) σ-formula φ if it appears in φ but is not quantified in φ. For a vector $\vec{v} := (v_{n_1}, v_{n_2}, \dots, v_{n_k})$, we call $\varphi(\vec{v})$ an *extended σ-formula* if all free variables of φ appear in $\vec{v}$ and none of the quantified variables of φ appear in $\vec{v}$. Note that as with extended σ-terms, $\vec{v}$ may contain extra variables that don't appear in φ.
A σ-formula is called a *sentence* if it has no free variables.

Notation regarding relations. For a k-ary relation R on a set X, i.e. $R \subseteq X^n$, and $\vec{a} \in X^n$, we often write "$R(\vec{a})$" or "$R(\vec{a})$ holds" to just mean that $\vec{a} \in R$. We also say that "$R(\vec{a})$ fails" if $\vec{a} \notin R$.
For example, for a binary relation $\leq$, we write $x_1 \leq x_2$ and not $(x_1, x_2) \in \leq$.
A 0-ary relation R on X is a subset of $X^0 := \{\emptyset\}$. Thus, either $R = X^0$ (always true) or $R = \emptyset$ (always false).

Def. For a σ-structure $\underline{A} := (A, \sigma)$ and an extended σ-formula $\varphi(\vec{v})$ with $n := |\vec{v}|$, we define the interpretation of $\varphi^{\underline{A}}(\vec{v})$ in $\underline{A}$ as an n-ary relation by induction on the construction/length of φ as follows: for all $\vec{a} \in A^n$:

(i) $\varphi := t_1 = t_2$. Then it must be that $t_1(\vec{v})$ and $t_2(\vec{v})$ are extended σ-terms and we define $\varphi^{\underline{A}}(\vec{v})(\vec{a}) \;:\Longleftrightarrow\; t_1^{\underline{A}}(\vec{a}) = t_2^{\underline{A}}(\vec{a})$.

(ii) $\varphi := R(t_1, \dots, t_k)$, where $R \in Rel_k(\sigma)$, $t_1, \dots, t_k$ are σ-terms. Then again it must be that $t_i(\vec{v})$ is an extended σ-term for $i = 1, \dots, k$, so we define

$$\varphi^{\underline{A}}(\vec{v})(\vec{a}) \;:\Longleftrightarrow\; R^{\underline{A}}(t_1^{\underline{A}}(\vec{a}), \dots, t_k^{\underline{A}}(\vec{a})).$$

(iii) $\varphi := \psi_1 \vee \psi_2$, for σ-formulas ψ_1, ψ_2. Then it must be that $\psi_1(\vec{v})$ and $\psi_2(\vec{v})$ are extended σ-formulas, and we define by induction:

$$\varphi^{\underline{A}}(\vec{v})(\vec{a}) \;:\Longleftrightarrow\; \psi_1^{\underline{A}}(\vec{v})(\vec{a}) \text{ or } \psi_2^{\underline{A}}(\vec{v})(\vec{a}).$$

(i.e. the union of $\psi_1^{\underline{A}}(\vec{v})$ and $\psi_2^{\underline{A}}(\vec{v})$).

(iv) $\varphi := \neg\psi$, for a σ-formula ψ. Then $\psi(\vec{v})$ is an extended σ-formula so we define by induction:

$$\varphi^{\underline{A}}(\vec{v})(\vec{a}) \;:\Longleftrightarrow\; \psi^{\underline{A}}(\vec{v})(\vec{a}) \text{ fails.}$$

(i.e. the complement of $\psi^{\underline{A}}(\vec{v})$)

(v) $\varphi := \exists u\, \psi$, where u is a variable and ψ is a σ-formula. Then u does not appear in $\vec{v}$ and is not quantified in ψ. Thus, $\psi(\vec{v}, u)$ is an extended σ-formula and we define by induction:

$$\varphi^{\underline{A}}(\vec{v})(\vec{a}) \quad :\Longleftrightarrow \text{ there is } b \in A \text{ such that } \psi^{\underline{A}}(\vec{a}, b) \text{ holds.}$$

Caution. As (v) shows, we may only quantify over the elements of the underlying structure. This why we say that we study first-order logic. The second-order logic allows quantification over subsets of the underlying structure, and is beyond this course.

Notation. Another way of saying that $\varphi^{\underline{A}}(\vec{v})(\vec{a})$ holds is to say that $\underline{A}$ satisfies $\varphi(\vec{v})(\vec{a})$, and we write this as $\underline{A} \models \varphi(\vec{v})(\vec{a})$.

Like with terms, we also drop $(\vec{v})$ from notation and simply write $\underline{A} \models \varphi(\vec{a})$ if $\vec{v}$ is clear from the context. As above, we will use other logical connectives $\wedge, \rightarrow, \longleftrightarrow$ and quantifier $\forall$ as abbreviations for $\neg(\neg \vee \neg)$, $(\neg\varphi \vee \psi)$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, $\neg \exists u \neg \psi$.

We also write $t_1 \neq t_2$ for $\neg(t_1 = t_2)$.

Examples. (a) $\sigma_{\text{arithm}} := (0, S, +, \cdot)$, where S is a unary function symbol and the other symbols are as expected. Recall that $\dot{2} := S(S(0))$ and let $\varphi := \dot{2} = v_0$. Then $\varphi(v_0)$ is an extended σ_{arithm}-formula and is interpreted in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, where $S^{\underline{N}} : \mathbb{N} \to \mathbb{N},\ n \mapsto n+1$, is a unary relation where

$$\varphi^{\underline{N}}(v_0)(a) \text{ holds} \iff a = 2,$$

i.e. $\varphi^{\underline{N}}(v_0) = \{2\} \subseteq \mathbb{N}$

(b) Again in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, we have:

- for $\leq(x,y) := \exists z\, (x + z = y)$, $\underline{N} \models \leq(a,b) \iff a \leq b$, for all $(a,b) \in \mathbb{N}^2$.

- for $div(x,y) := \exists u(x \cdot u = y)$, $\underline{N} \models div(a,b) \Leftrightarrow a$ divides b, for all $(a,b) \in \mathbb{N}^2$.
- for $prime(y) := \forall x(div(x,y) \to (x = 1 \vee x = y))$, $\underline{N} \models prime(a) \Leftrightarrow a$ is prime, for all $a \in \mathbb{N}$.
- for each fixed natural number $n \in \mathbb{N}$, we can write a σ_{arith}-formula $Fermat_n := \forall x \forall y \forall z((\underbrace{x \cdot x \cdot \ldots \cdot x}_{n} + \underbrace{y \cdot y \cdot \ldots \cdot y}_{n} = \underbrace{z \cdot z \cdot \ldots \cdot z}_{n}) \to (x = 0 \vee y = 0 \vee z = 0))$.
  Then we know that $\underline{N} \models \neg Fermat_2$ because $3^2 + 4^2 = 5^2$ for example, but due to the famous of A. Wiles, we now know that $\underline{N} \models Fermat_n$ for all $n \geq 3$.

Caution. It seems like we can't define the exponentiation function in this structure, i.e. a σ_{arith}-formula $\varphi(v_0, v_1)$ such that $\underline{N} \models \varphi(a,b,c) \Leftrightarrow a^b = c$, so we didn't use this. However, it is true that such a φ exists, but is difficult to see why. We will leave it as an exercise.

- $Goldbach := \forall x(div(2,x) \to \exists y \exists z(Prime(y) \wedge Prime(z) \wedge (x = y + z)))$.
  Its interpretation in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$ is the famous Goldbach conjecture, which is still one of the widest open problems in number theory, so we don't know whether $\underline{N} \models Goldbach$ or not.

(c) For the same formula $\varphi := \dot{2} = v_0$, $\varphi(v_0, v_1)$ is an extended σ_{arith}-formula and its interpretation in $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$ is a binary relation $\varphi^{\underline{N}}(v_0, v_1) \subseteq \mathbb{N}^2$ such that for all $(a,b) \in \mathbb{N}^2$,

$$\underline{N} \models \varphi(a,b) \Leftrightarrow a = 2.$$

(d) In $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot)$, the extended formula $Pos(x) := \exists y(x = y \cdot y)$. Then for all $a \in \mathbb{R}$, we have $\underline{R} \models Pos(a) \Leftrightarrow a$ is non-negative.